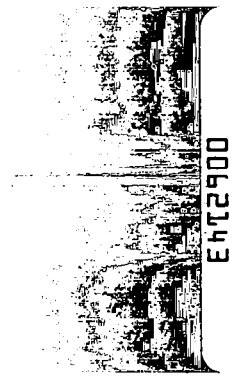

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Finite-Difference Solution of the Compressible Stability Eigenvalue Problem

Mujeeb R. Malik

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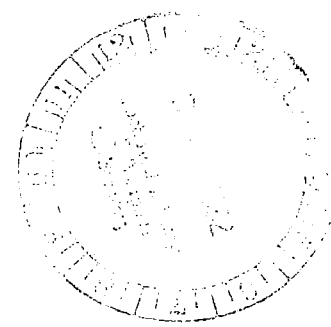

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Finite-Difference Solution of the Compressible Stability Eigenvalue Problem

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Prepared for
Langley Research Center
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SUMMARY

A compressible stability analysis computer code is developed. The code uses a matrix finite-difference method for local eigenvalue solution when a good guess for the eigenvalue is available and is significantly more computationally efficient than the commonly used initial-value approach. The local eigenvalue search procedure also results in eigenfunctions and, at little extra work, group velocities. A globally convergent eigenvalue procedure is also developed which may be used when no guess for the eigenvalue is available. The global problem is formulated in such a way that no unstable spurious modes appear so that the method is suitable for use in a black-box stability code. Sample stability calculations are presented for the boundary layer profiles of an LFC swept wing.

SECTION 1 - INTRODUCTION

The stability properties of compressible laminar boundary layers are particularly relevant to the phenomenon of laminar-turbulent flow transition. Recently, interest in this problem has increased because of applications to Laminar Flow Control (LFC) technology. In such applications there is a need for fast computer codes to perform efficient design calculations. The computer code SALLY [1] was developed for this purpose. It can perform optimized stability calculations for determining suction requirements for maintaining laminar flow over swept wings. However, SALLY uses incompressible stability theory so it solves the eigenvalue problem for the fourth-order Orr-Sommerfeld differential equation.

The linear stability analysis of three-dimensional compressible boundary layers involves solution of an eigenvalue problem for an eighth order system of differential equations. In the case of two-dimensional boundary layers or in the absence of dissipation in three-dimensional flow, the eighth-order system reduces to sixth order.

The basic equations for the linear stability analysis of parallel-flow compressible boundary layers are derived using the samll disturbance theory. Infinitesimal disturbances of sinusoidal form are imposed on the steady boundary layer flow and substituded in the compressilbe Navier-Stokes equations.

Assuming that the mean flow is locally parallel, a set of five ordinary differential equations is obtained. Of these, there are three second order momentum equations, one second order energy equation and one first order continuity equation. Following Lin [2-4] (his work involved only sixth order system), this system is reduced to a set of eight first order ordinary differential equations making the system amenable to initial-value numerical integration procedures.

All previous numerical investigations of compressible flow stability [5-10] make use of the initial value approach for the solution of this system of eight first-order equations (or the reduced system of six first-order equations). In these studies, the integration is started at a boundary (usually the free stream boundary since the asymptotic solution of the stability equations provides initial values for starting the integration) and marched to the other boundary (solid wall) typically using a Runge-Kutta integration procedure. Four linearly independent solutions are sought by means of this integration. The difficulty encountered in this scheme is that a straight-forward integration fails to produce four independent solutions because of parasitic error growth. This difficulty is usually overcome by an orthonormalization technique [see, e.g., 11]. Upon obtaining four accurate linearly independent solutions, linear combinations are formed to satisfy all but one boundary condition at the wall (the boundary towards which the equations are marched). The remaining boundary condition is satisfied only

when the eigenvalue condition is satisfied. It is normally imposed iteratively using a Newton-Raphson procedure which results in the desired eigenvalue.

These initial-value methods to solve the compressible stability equations are often computationally slow and thus inefficient if used in a black-box stability code. Furthermore, the initial-value methods require a good guess for the eigenvalue which is usually not available when one encounters a new problem. In such cases one can obtain eigenvalues using a local search procedure only by trial and error, which is neither elegant nor efficient.

In the present work a computer code is developed for the compressible stability analysis of three-dimensional boundary layers. The code includes two eigenvalue search procedures—global (which is to be used when no guess is available) and local (which is used when a good guess is available). The stability equations are solved in their original (3 second-order momentum equations, one second-order energy and one first-order continuity equation) using a matrix finite-difference method. The reduction of the normal momentum equation to a first order equation for pressure fluctuations as done by Lees and Lin [2] is not done here since that results in unstable spurious modes when the problem is solved using the global method (see below).

The finite-difference representation of the stability equations results in a block-tridiagonal system of equations. A generalized matrix eigenvalue problem is then set up and solved using the complex LR algorithm [12] for global eigenvalue search. The local search is performed by inverting the block-tridiagonal system using a block LU factorization together with inverse Rayleigh iteration for the eigenvalues [12] in which the eigenvalue, eigenfunction and its adjoint are obtained simultaneously. One can obtain the group velocities which are needed in an eigenvalue optimization procedure [13] at little extra cost to the local eigenvalue search. The procedure used in the present study for local eigenvalue solution is significantly faster than the initial value approach employed by previous investigators.

Some sample calculations are performed for the stability analysis of compressible three-dimensional boundary layer profiles obtained for a laminar flow control wing using the boundary layer code developed by Kaups & Cebeci (see ref. [1]). particular wing chosen is a 35° swept super-critical wing with an 8 foot chord. The free stream Mach number is 0.89. Mack [7] and Lekoudis [9] reported stability calculations for the same wing using their stability codes. We have chosen three stations on the wing which represent forward and rearward crossflow instability regions and a midchord streamwise instability region. Some of the relevant boundary data is given in Table 1. More details on the wing and pressure and suction distributions are given in reference [7] and [9].

SECTION 2 - COMPRESSIBLE STABILITY EQUATIONS

Consider the stability of a three-dimensional locally-parallel compressible boundary layer flow. Let us use a cartesian coordinate system x,y,z in which the y-axis is normal to the solid boundary and x,z are parallel to it. (In the particular case of a wing, the x-axis will be assumed to be in the direction of the normal chord and the z-axis will be along the span.) If u,v,w are the x,y,z components of the instantaneous velocity, respectively, and p and τ are instantaneous pressure and temperature, then, assuming that the base flow is locally parallel,

$$u(x,y,z,t) = U_o(y) + \tilde{u}(y)e^{i(\alpha x+\beta z-\omega t)} \tag{1}$$

$$v(x,y,z,t) = \tilde{v}(y)\ e^{i(\alpha x+\beta z-\omega t)} \tag{2}$$

$$w(x,y,x,t) = W_o(y) + \tilde{w}(y)e^{i(\alpha x+\beta z-\omega t)} \tag{3}$$

$$p(x,y,z,t) = \tilde{p}(y)\ e^{i(\alpha x+\beta z-\omega t)} \tag{4}$$

$$\tau(x,y,z,t) = T_o(y) + \tilde{\tau}(y)e^{i(\alpha x+\beta z-\omega t)} \tag{5}$$

Here U_o, W_o and T_o represent the steady unperturbed boundary layer and quantities with tildas denote complex disturbance amplitudes.

Substituting Eqs. (1-5) into the compressible Navier-Stokes equations, it can be shown that the linear parallel disturbances satisfy the following system of ordinary differential equations:

$$(A\,D^2 + B\,D + C)\vec{\phi} = 0 \qquad (6)$$

where $\vec{\phi}$ is a five-element vector defined by

$$\{\alpha\tilde{u} + \beta\tilde{w},\ \tilde{v},\ \tilde{p},\ \tilde{\tau},\ \alpha\tilde{w} - \beta\tilde{u}\}^T \qquad (7)$$

Here A,B,C are 5 x 5 matrices, and $D \equiv \frac{d}{dy}$, where y is the normal boundary layer coordinate. The non-zero elements of the matrices A,B,C are given in Appendix I.

The boundary conditions for Eq. (6) are

$$y = 0;\ \phi_1 = \phi_2 = \phi_4 = \phi_5 = 0 \qquad (8)$$

$$y \to \infty;\ \phi_1,\ \phi_2,\ \phi_4,\ \phi_5 \to 0$$

Equations (6-8) constitute an eigenvalue problem for the frequency ω and wavenumbers α,β. For a given Reynolds number R, this eigenvalue problem provides a complex dispersion relation of the form

$$\omega = \omega(\alpha,\beta) \qquad (9)$$

relating the parameters α,β and ω which, in general, are all complex. In our analysis we choose to use temporal stability theory in which α,β are real and ω is complex. It is thus assumed that the wavelike disturbances have x and z components of wave number α and β, respectively, and have a frequency $\omega_r = \mathrm{Re}(\omega)$. It is further assumed that the disturbances grow or decay only in time. They grow if

$\omega_i = \mathrm{Im}(\omega) > 0$ and decay if $\omega_i < 0$.

The selection of temporal theory results in a linear eigenvalue problem for ω so that global eigenvalue techniques can be applied. It is possible to extend the global method to the spatial problem for the compressible stability equations but the work involved and the computer resources required discourage any such attempt. It appears that the most efficient way to automatically provide a guess for the local solution of the spatial stability problem is to first solve the temporal global problem and then obtain the spatial guess using the group velocity transformations [14].

Equation (6) represents three second order momentum equations, one second-order energy equation and one first order continuity equation. It is possible to eliminate pressure entirely between these equations and to reduce the system to four second-order equations for velocity and temperature fluctuations. This reduction may seem desirable for the numerical integrations since the order of the matrix will be reduced. However, the elimination of pressure makes the equations singular and their solution becomes difficult to obtain in a numerically stable way. Similarly, it is possible to reduce the system of eight first-order equations to a single eighth-order differential equation [15], but this too is computationally ill conditioned. Thus, the following analysis is carried out in terms of five equations: three second-order equations for streamwise and spanwise velocity and for energy, one first-order equation for continuity and one

second order (but effectively first-order) equation for the normal velocity component.

SECTION 3 - FINITE-DIFFERENCE REPRESENTATION

The governing system of equations (6) for compressible flow is represented using a second order finite-difference formula on a staggered mesh (see Figure 1).

First the boundary layer coordinate y, $0 \leq y \leq y_e$ is mapped into the computational domain $0 \leq \eta \leq 1$ by the algebraic mapping

$$\eta = \frac{g\, y}{L+y} \tag{10}$$

where $g = 1 + \frac{L}{y_e}$

Here y_e is the location of the edge of the boundary layer and L is a scaling parameter chosen to optimize the accuracy of the calculations. After some experience it has been found that a good choice for L is $L \simeq 2y_o$ where y_o is the value of y at which the streamwise component of the mean velocity, U_o, achieves 1/2 its freestream value. In the present study we choose $y_e = 100$. However, a much smaller value of $y_e (\simeq 15)$ can be chosen for the local method since the free stream boundary conditions are imposed using the asymptotic solution.

The computational domain η is divided into N equal intervals and the second order equations are represented as

$$f_1 A_j \left[\frac{\vec{\phi}_{j+1} - 2\vec{\phi}_j + \vec{\phi}_{j-1}}{\Delta\eta} \right]$$

$$+ d_1 [(f_2 A_j + f_3 B_j) \left(\frac{\vec{\phi}_{j+1} - \vec{\phi}_{j-1}}{2\,\Delta\eta} \right) + C_j \vec{\phi}_j]$$

$$+ d_2 \left[f_3 B_j \left(\frac{\vec{\phi}_{j+\frac{1}{2}} - \vec{\phi}_{j-\frac{1}{2}}}{\Delta\eta} \right) + C_j \left(\frac{\vec{\phi}_{j+\frac{1}{2}} + \vec{\phi}_{j-\frac{1}{2}}}{2} \right) \right] = 0$$

$$(j=1,\ldots,N-1) \quad (11)$$

where ϕ_j is the value of $\vec{\phi}$ at $\eta = j/N$ and has components $\phi_{k,j}$ $(k = 1,\ldots,5)$. Also,

$d_1 = 1$, $d_2 = 0$ except

$d_1 = 0$, $d_2 = 1$ for the $\tilde{p}$ component of $\vec{\phi}$,

$$f_1 = \frac{(g-\eta)^4}{g^2 L^2}$$

$$f_2 = -\frac{2(g-\eta)^3}{g^2 L^2}$$

$$f_3 = \frac{(g-\eta)^2}{gL}$$

The first order continuity equation is represented as

$$f_3 B_{j+\frac{1}{2}} \frac{\vec{\phi}_{j+1} - \vec{\phi}_j}{\Delta\eta} + C_{j+\frac{1}{2}} \vec{\phi}_{j+\frac{1}{2}} = 0 \quad (12)$$

$$(j = 0,\ldots,N-1)$$

Equations (11,12) along with the 8 boundary conditions (8) represent 5N+4 equations for 5N+4 unknowns. Since the velocity and temperature disturbances are assumed to be identically zero at the solid boundary ($\eta = 0$) the system reduces to 5N equations for 5N unknowns when these boundary conditions are applied. This is a block-tridiagonal system

of equations with 5 x 5 blocks which is solved using fully pivoted LU factorization.

In the global method, the free-stream conditions ($\eta = 1$) are that $\phi_1=\phi_2=\phi_4=\phi_5=0$ at $\eta = 1$. This results in a linear eigenvalue problem for η. In the local method, asymptotic behavior of $\vec{\phi}$ as $y \to \infty$ is found from (6) (see appendix II). This asymptotic behavior is used to obtain a free-stream boundary condition of the form

$$(ED + F)\ \vec{\phi} = 0 \qquad (13)$$

that is applied at $\eta = 1$ on components $\phi_k (k=1,2,4,5)$.

SECTION 4 - GLOBAL METHOD

When no guess is available for the eigenvalue of interest, it is best to use a method that is globally convergent and nearly guaranteed to converge to the eigenvalue. Such a method may be based on algorithms for calculations of the eigenvalues of a general complex matrix [12].

When the compressible stability equations (11-12) are formulated as a matrix problem, they take the form

$$\bar{A}\,\vec{\phi} = \omega\,\bar{B}\,\vec{\phi} \tag{14}$$

where ω is the eigenvalue and $\vec{\phi}$ is the discrete representation of the eigenfunction. The eigenvalue is determined by the determinant condition

$$\mathrm{Det}\left|\,\bar{A} - \omega\,\bar{B}\right| = 0 \tag{15}$$

For the present problem, $\bar{B}$ is invertible so (15) may be solved as

$$\mathrm{Det}\left|\,\bar{B}^{-1}\,\bar{A} - \omega\,I\right| = 0 \tag{16}$$

which is the standard matrix eigenvalue problem, solvable by LR methods [12].

One has to be very careful in formulating the eigenvalue problem (14) using global methods to avoid the generation of growing (unstable) modes that are not physically relevant. These spurious unstable modes do not correspond to solutions of the differential equation— as the spatial resolution used to discretize the eigenfunction increases true modes of the differential equation converge while spurious modes do not.

A clumsy way to distinguish spurious modes from true modes is to change the spatial resolution and retain only those modes that do not change appreciably. This is neither efficient nor elegant.

A better way is to eliminate the spurious unstable modes entirely. Spurious stable modes are still possible, but since these stable modes are normally very stable, they are not of much interest and can be easily disregarded without testing their true nature. We shall now describe a technique for eliminating the spurious unstable modes.

The idea is simply that the spurious unstable modes would, if we used the same numerical method used for the stability problem on an initial-value problem instead, cause the unconditional instability of the numerical solution of the initial-value problem. On the other hand, if we were careful enough to use a numerical method for the stability problem that was also numerically stable for the initial-value problem, then no spurious unstable modes would exist.

Thus, one way to avoid spurious modes is to set up the problem as one would for an initial value problem and to use a finite-difference scheme for the eigenvalue analysis that is consistent with the scheme for the initial-value problem. This is the method used in the present study; no spurious unstable modes appear in our calculations. In some initial work on this problem, we followed Lees and Lin [2] by reducing the second-order normal momentum equation to a first-order equation for pressure using the continuity equation.

This gives three second-order equations (two momentum and one energy equation) and two first-order equations (one for pressure and another for normal velocity). When the matrix eigenvalue problem is set up using this system of equations by means of finite-differencing, several unstable spurious modes appear.

When the eigenvalue problem (16) is solved using the LR algorithm, the storage requirements are of $0(K^2)$ while the computational work involved is of $0(K^3)$ where $K = 5N$ for the eighth order system of equations and $K = 4N$ for the sixth-order system. It can be seen that the global method is relatively expensive computationally so it should be only used when no guess of the eigenvalue is available.

Some computer timings for the global method on a CYBER 175 computer are given in Table 2. All timings were obtained using the internal clock and are averaged over the three test cases listed in Table 1. Since the eigenvalue obtained by the sixth-order system is not much different from that of the eighth-order system, the use of the sixth-order system for the global method is recommended.

SECTION 5 - LOCAL EIGENVALUE SEARCH

If a guess for the eigenvalue is available then one can improve its value by a local method. One way to perform the local analysis is to use a simple iterative method to find the eigenvalues of the matrix equation (15) that approximates the compressible stability equations. An effective and efficient procedure for doing this is to use the inverse Rayleigh iteration procedure [12]. The generalization of this procedure to the compressible stability problem results in the following algorithm

$$(\bar{A} - \omega_k \bar{B})\ \vec{\phi}^{(k+1)} = \bar{B}\ \vec{\phi}^{(k)} \tag{17}$$

$$(\bar{A} - \omega_k \bar{B})^T\ \vec{\psi}^{(k+1)} = \bar{B}^T\ \vec{\psi}^{(k)} \tag{18}$$

$$\omega_{k+1} = \frac{(\vec{\psi}^{(k+1)}, \bar{A}\ \vec{\phi}^{(k+1)})}{(\vec{\psi}^{(k+1)}, \bar{B}\ \vec{\phi}^{(k+1)})} \tag{19}$$

The iteration cycle is started with a guessed eigenvalue ω_o and by assuming any smooth functional distribution for the eigenfunction $\vec{\phi}(0)$ and its adjoint $\vec{\psi}(0)$. In an integration of stability characteristics across a wing $\vec{\phi}(0)$ and $\vec{\psi}(0)$ may be chosen as the eigenfunction and adjoint from the previous station.

At the end of each iteration cycle k the eigenfunction and its adjoint are normalized so that

$$\vec{\phi}^{(k+1)} = \vec{\phi}^{(k+1)}/\max(\vec{\phi}^{(k+1)})$$
$$\vec{\psi}^{(k+1)} = \vec{\psi}^{(k+1)}/\max(\vec{\psi}^{(k+1)}) \qquad (20)$$

The algorithm (17-20) has a rapid (cubic) rate of convergence. The error satisfies $\omega_{k+1} - \omega = 0((\omega_k - \omega)^3)$.

We solve the block-tridiagonal form of Eq. (17) by using a fully pivoted LU method, in which case the same LU factorization applies to the solution of the adjoint problem (18).

In practice it is not necessary (rather, not recommended) to update the eigenvalue approximation ω_k after each iteration to the eigenfunction and its adjoint using (17-18). We have found it to be most efficient to iterate (17-18) approximately 4-10 times while keeping ω_k fixed (and, therefore, using the same LU factorization of $\bar{A}-\omega_k\bar{B}$). Once the eigenfunction is refined less than 5 iterations are required for a fixed ω_k. Generally, only two outer iterations [LU factorizations and applications of (19) to update ω_k to ω_{k+1}] are required to converge to an eigenvalue. Some data on the speed of the local eigenvalue solution is given in Table 3. The time reported also includes the calculation of group velocity. Since the eigenfunction and its adjoint are available as a result of local eigenvalue search, it costs little extra work to compute group velocity (see below).

To show the sensitivity of the local method to the initial guess some sample calculations are made and reported in Table 4. The converged eigenfunction distributions vs y for case 1 using 80 points are plotted in Figure 2.

SECTION 6 - CALCULATION OF GROUP VELOCITY

The group velocity is of importance in relating the results of spatial and temporal stability theory and in several optimization problems [13]. In a layered flow with three-dimensional disturbances having wavevector (α,β) and frequency $\omega(\alpha,\beta)$, the group velocity $\vec{v}_g$ is

$$\vec{v}_g = \left(\frac{\partial\omega}{\partial\alpha}\ ,\ \frac{\partial\omega}{\partial\beta}\right) \tag{21}$$

One way to compute the group velocity is simply to compute the frequency ω for several nearby values of α,β and then use finite-difference approximations to $\vec{v}_g$. However, this procedure is not very efficient.

A much better way to compute the group velocity is to first write the compressible stability equations for three dimensional disturbances in the form

$$L(\alpha,\beta,\omega(\alpha,\beta))\vec{\phi} = 0 \tag{22}$$

Taking the derivative of (22) with respect to α gives

$$\frac{\partial L}{\partial\alpha}\vec{\phi} + \frac{\partial L}{\partial\omega}\ \frac{\partial\omega}{\partial\alpha}\vec{\phi} + L\ \frac{\partial\vec{\phi}}{\partial\alpha} = 0 \tag{23}$$

Taking the inner product of (23) with the adjoint $\vec{\psi}$ of the eigenfunction $\vec{\phi}$ gives

$$\frac{\partial\omega}{\partial\alpha} = -\frac{(\vec{\psi},\frac{\partial L}{\partial\alpha}\vec{\phi})}{(\vec{\psi},\frac{\partial L}{\partial\omega}\vec{\phi})} \tag{24}$$

since $(\vec{\psi}, L\frac{\partial\vec{\phi}}{\partial\alpha}) = (L^T\vec{\psi}, \frac{\partial\vec{\phi}}{\partial\alpha}) = 0$. There is a similar expression for $\frac{\partial\omega}{\partial\beta}$. Note that the inner product in (24) is the usual L_2 vector inner product:

$$(\vec{f},\vec{g}) = \sum_{i=1}^{N} f_i\, g_i \qquad (25)$$

It is not necessary to use either the adjoint eigenfunction of the differential equations (6) or the inner product $\int f(y)g(y)dy$; $\vec{\psi}$ is the adjoint of $\vec{\phi}$ with respect to the simple discrete inner product (25) and suffices to annihilate the term $\partial\vec{\phi}/\partial\alpha$ in (23).

The computation of the group velocity using (24) requires only the calculation of $\partial L/\partial\alpha\ \vec{\phi}$ (and $\partial L/\partial\beta\ \vec{\phi}$) since $\vec{\psi}$, $\vec{\phi}$, $\partial L/\partial\omega\ \vec{\phi}$ are available from the local eigenvalue iterations.

In Table 5, group velocities calculated by the present method for the three test cases are compared with those calculated by central differencing:

$$\left.\frac{\partial\omega}{\partial\alpha}\right|_j = \frac{\omega_{j+1}-\omega_{j-1}}{\alpha_{j+1}-\alpha_{j-1}} \qquad (26)$$

$$\left.\frac{\partial\omega}{\partial\beta}\right|_j = \frac{\omega_{j+1}-\omega_{j-1}}{\beta_{j+1}-\beta_{j-1}}$$

which required four extra eigenvalue calculations so the cost of determining group velocity was four times the cost of the eigenvalue search. Using the present method, group

velocity can be obtained at less than 10% of the cost of the eigenvalue search.

SECTION 7 - RICHARDSON EXTRAPOLATION

The finite difference method presented above is only second order accurate. However, the accuracy of the eigenvalue (and the group velocity) obtained by this procedure can be enchanced by Richardson extrapolation to the limit $\Delta\eta = 1/N \to 0$ [15].

If

$$S_i^{(0)} = \omega(h_i) \quad (i = 0,\ldots m) \tag{27}$$

where h_i is the grid size and ω is the eigenvalue computed by a method with error of $0(h^2)$, then

$$S_i^{(j)} = S_{i+1}^{(j-1)} + \frac{S_{i-1}^{(j-1)} - S_i^{(j-1)}}{\left(\frac{h_i}{h_{i+j}}\right)^2 - 1} \tag{28}$$

$$(j=1,\ldots,m,\ i=0,\ldots,m-j)$$

The resulting approximation $S_i^{(j)}$ has an error of order $0(h^{j+2})$ when appropriate sequences of grid sizes h_i are used. We present here eigenvalue calculations for different grids ($h_i = \frac{1}{N_i}$) and their extrapolated values. The sequence of h_i chosen is that proposed by Bulirsch and Stoer [16]. Tables 6,7,8 give eigenvalue results for boundary layer data cases 1,2 and 3, respectively.

It can be seen that an eigenvalue converged to 5 significant digits is obtained as a result of the extrapolation procedure. For most applications, an eigenvalue that has converged to 3 significant digits should suffice. This is

achieved by extrapolating only between three eigenvalues at N = 20, 40 and 60. For three point extrapolation, (28) reduces to the following formula

$$\omega_{\text{extrapolated}} = \left\{[h_1^2-h_2^2)/h_0^2]\omega_0 + [(h_2^2-h_0^2)/h_1^2]\omega_1 + [(h_0^2-h_1^2)/h_2^2]\omega_2\right\} \Big/ \left\{ [(h_1^2-h_2^2)/h_0^2] + [(h_2^2-h_0^2)/h_1^2] + [(h_0^2-h_1^2)/h_2^2] \right\} \tag{29}$$

It is our experience that a reasonable answer can be obtained by this extrapolation formula using only 20, 30 and 40 points. Eigenvalues obtained using this set of points are compared with the converged value using 20, 40, 60, 80, 120 and 160 points in Table 9. The eigenvalue accurate to 3 significant digits is obtained in about 2 seconds of CYBER 175 time.

SECTION 8 - CONCLUSION

Efficient local and global eigenvalue methods for the stability analysis of plane-parallel three-dimensional compressible boundary layers have been developed. The computer code that implements these matrix boundary-value methods is significantly more efficient than previous codes based on initial-value methods and is suitable for a black-box stability analysis package.

APPENDIX I

The Stability Equations

The stability equations for three-dimensional compressible boundary layers are written as in (6):

$$(A\ D^2 + B\ D + C)\ \vec{\phi} = 0$$

where

$$\phi_1 = \alpha\ \tilde{u} + \beta\ \tilde{w}$$

$$\phi_2 = \tilde{v}$$

$$\phi_3 = \tilde{p}$$

$$\phi_4 = \tilde{\tau}$$

$$\phi_5 = \alpha\ \tilde{w} - \beta\ \tilde{u}$$

and

$$D \equiv \frac{d}{dy}$$

Here $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ represent the complex amplitudes of the perturbation velocity in x, y, z directions, respectively and $\tilde{p}$, $\tilde{\tau}$ are the perturbation amplitudes of pressure and temperature. The disturbances are assumed to be of the form

$$\vec{\phi}(y)\ e^{i(\alpha x + \beta z - \omega t)}$$

where α, β are x and z wave numbers, respectively and ω is the (complex) frequency.

The non-zero elements of 5x5 matrices A, B and C are given by

$$A_{11} = 1$$

$$A_{22} = 1$$

$$A_{44} = 1$$

$$A_{55} = 1$$

$$B_{11} = \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} T_o'$$

$$B_{12} = i(\lambda-1)\ (\alpha^2 + \beta^2)$$

$$B_{14} = \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} (\alpha U_o' + \beta W_o')$$

$$B_{21} = i(\lambda-1)/\lambda$$

$$B_{22} = \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} T_o'$$

$$B_{23} = -\frac{R}{\mu_o \lambda}$$

$$B_{32} = 1$$

$$B_{41} = 2(\gamma-1)M^2\ \sigma\ (\alpha U_o' + \beta W_o')/(\alpha^2 + \beta^2)$$

$$B_{44} = \frac{2}{\mu_o} \frac{d\mu_o}{dT_o} T_o'$$

$$B_{45} = 2(\gamma-1)M^2\ \sigma\ (\alpha W_o' - \beta U_o')/(\alpha^2 + \beta^2)$$

$$B_{54} = \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} (\alpha W_o' - \beta U_o')$$

$$B_{55} = \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} T_o'$$

$$C_{11} = -[\frac{i\ R}{\mu_o T_o} (\alpha U_o + \beta W_o - \omega) + \lambda(\alpha^2 + \beta^2)]$$

$$C_{12} = -[\frac{R}{\mu_o T_o} (\alpha U_o' + \beta W_o') - \frac{i}{\mu_o} \frac{d\mu_o}{dT_o} T_o' (\alpha^2 + \beta^2)]$$

$$C_{13} = -\frac{i\ R}{\mu_o} (\alpha^2 + \beta^2)$$

$$C_{14} = (\alpha U_o' + \beta W_o') \frac{1}{\mu_o} \frac{d^2\mu_o}{dT_o^2} T_o' + (\alpha U_o'' + \beta W_o'') \frac{1}{\mu_o} \frac{d\mu_o}{dT_o}$$

$$C_{21} = i \frac{\lambda-2}{\lambda\mu_o} \frac{d\mu_o}{dT_o} T_o'$$

$$C_{22} = -[\frac{i\ R}{\mu_o T_o \lambda} (\alpha U_o + \beta W_o - \omega) + (\alpha^2 + \beta^2)/\lambda]$$

$$C_{24} = \frac{i}{\lambda} \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} (\alpha U_o' + \beta W_o')$$

$$C_{31} = i$$

$$C_{32} = - \frac{T_o'}{T_o}$$

$$C_{33} = i \gamma M^2 (\alpha U_o + \beta W_o - \omega)$$

$$C_{34} = - \frac{i}{T_o} (\alpha U_o + \beta W_o - \omega)$$

$$C_{42} = - [\frac{R \sigma}{\mu_o T_o} T_o' - 2 i (\gamma-1) M^2 \sigma (\alpha U_o' + \beta W_o')]$$

$$C_{43} = \frac{i R \sigma}{\mu_o} (\gamma-1) M^2 (\alpha U_o + \beta W_o - \omega)$$

$$C_{44} = - [\frac{i R \sigma}{\mu_o T_o} (\alpha U_o + \beta W_o - \omega) + (\alpha^2 + \beta^2)$$
$$- (\gamma-1) \sigma M^2 \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} (U_o'^2 + W_o'^2)$$
$$- \frac{1}{\mu_o} \frac{d^2\mu_o}{dT_o^2} (T_o')^2 - \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} T_o'']$$

$$C_{52} = - \frac{R}{\mu_o T_o} (\alpha W_o' - \beta U_o')$$

$$C_{54} = \frac{1}{\mu_o} \frac{d^2\mu_o}{dT_o^2} T_o' (\alpha W_o' - \beta U_o') + \frac{1}{\mu_o} \frac{d\mu_o}{dT_o} (\alpha W_o'' - \beta U_o'')$$

$$C_{55} = - [\frac{i R}{\mu_o T_o} (\alpha U_o + \beta W_o - \omega) + \alpha^2 + \beta^2)]$$

Here subscript o refers to the unperturbed boundary layer and primed quantities represent the derivative of the quantity with respect to the boundary layer coordinate y.

In the above, μ and γ are the dynamic viscosity and the ratio of the specific heats respectively for the gas which is assumed to be perfect. The Prandtl number σ is assumed to be constant. Moreover, λ is defined as

$$\lambda = \frac{2}{3} (\mu_2 + 2)$$

where μ_2 is the ratio of the second coefficient of viscosity to the first.

In the present study all velocities have been scaled by U_e, the x component of velocity at the edge of the boundary layer, and all lengths are scaled by δ^*, the displacement thickness of the velocity profile in x-direction, $U_o(y)$. The resultant Reynolds number and Mach number are then given by

$$R = \frac{U_e \delta^*}{\nu_e}$$

$$M = \frac{U_e}{\sqrt{\gamma \Delta T_e}}$$

where ν_e and T_e are the kinematic viscosity and mean temperature in the free stream, and Δ is the universal gas constant.

The results reported in the present paper were obtained with $\sigma = 0.72$ and $\mu_2 = 1.2$. Dynamic viscosity, μ, was calculated using Sutherland's viscosity law. At least for the test cases considered in the present work, the results are

not sensitive to the value of μ_2.

The special choice of $\phi_1 = \alpha\tilde{u} + \beta\tilde{w}$ and $\phi_5 = \alpha\tilde{w} - \beta\tilde{u}$ allows the eighth-order system of equations to reduce to the sixth order for eigenvalue calculation if B_{45} is assumed to be zero. This assumption implies the absence of dissipation.

APPENDIX II

BOUNDARY CONDITIONS

In the free stream ($y \geq y_e$), the governing equations (6) reduce to

$$D^2\phi_1 + B_{12}\, D\, \phi_2 + C_{11}\, \phi_1 + C_{13}\, \phi_3 = 0 \qquad \text{(II-1)}$$

$$D^2\phi_2 + B_{21}\, D\, \phi_1 + B_{23}\, D\, \phi_3 + C_{22}\, \phi_2 = 0 \qquad \text{(II-2)}$$

$$D\phi_2 + C_{31}\, \phi_1 + C_{33}\, \phi_3 + C_{34}\, \phi_4 = 0 \qquad \text{(II-3)}$$

$$D^2\phi_4 + C_{43}\, \phi_3 + C_{44}\, \phi_4 = 0 \qquad \text{(II-4)}$$

$$D^2\phi_5 + C_{55}\, \phi_5 = 0 \qquad \text{(II-5)}$$

where $D \equiv \frac{d}{dy}$. The non-zero coefficients B_{ij} and C_{ij} are the same as in appendix I except that they are no more dependent on y. By eliminating ϕ_3, the above equations can be written as

$$D^2\psi_1 + a\, D\psi_2 + e\psi_1 + f\psi_3 = 0 \qquad \text{(II-6)}$$

$$D^2\psi_2 + b\, D\psi_1 + c\, D\psi_3 + g\psi_2 = 0 \qquad \text{(II-7)}$$

$$D^2\psi_3 + d\, D\psi_2 + s\psi_3 + h\psi_1 = 0 \qquad \text{(II-8)}$$

$$D^2\psi_4 + q\psi_4 = 0 \qquad \text{(II-9)}$$

where

$$\psi_1 = \phi_1,\ \psi_2 = \phi_2,\ \psi_3 = \phi_4,\ \psi_4 = \phi_5$$

and

$$a = B_{12} - C_{13}/C_{33}$$

$$b = (B_{21} - \frac{B_{23}C_{31}}{C_{33}})/(1 - \frac{B_{23}}{C_{33}})$$

$$c = (-B_{23}C_{34}/C_{33})/(1- \frac{B_{23}}{C_{33}})$$

$$d = -C_{43}/C_{33}$$

$$e = C_{11} - C_{13}\, C_{31}/C_{33}$$

$$f = -C_{13}\, C_{34}/C_{33}$$

$$g = C_{22}/(1-B_{23}/C_{33})$$

$$h = - \frac{C_{43}\, C_{31}}{C_{33}}$$

$$s = C_{44} - C_{43}\, C_{34}/C_{33}$$

$$q = C_{55}$$

Equations (II-6) - (II-9) admit a general solution of the form

$$\psi_i = \sum_{j=1}^{8} p_j\, z_i^{(j)}\, e^{\lambda_j (y-y_e)} \quad ;i=1,4 \qquad \text{(II-10)}$$

where λ_j are the eight characteristic roots of the coefficient matrix of Equations (II-6) - (II-9) and $z_i^{(j)}$ are the components of the eight characteristic vectors $z^{(j)}$.

The asymptotic condition

$$\psi_i \to 0 \quad \text{as} \quad y \to \infty$$

is imposed by requiring that the arbitrary constants p_j vanish when

$$\text{Re}(\lambda_j) > 0.$$

There are four λ_j which satisfy this condition resulting in four boundary conditions.

The roots λ_j (j = 1,...6) can be obtained from the characteristic equation (derived from Eqs. (II-6) - (II-8))

$$\lambda^6 + L_1 \lambda^4 + L_2 \lambda^2 + L_3 = 0$$

where

$$L_1 = g + e + k - ab - cd$$

$$L_2 = ke - hf + eg + kg - abk + bdf - ced + ahc$$

$$L_3 = keg - ghf$$

Assuming that $X = \lambda^2$, the above sixth order equation reduces to the cubic equation

$$X^3 + L_1 X^2 + L_2 X + L_3 = 0$$

The roots of this cubic equation can be found as in [17] and consequently six of the λ_j's are known. The components of the corresponding characteristic vectors are

$$z_1^{(j)} = 1$$

$$z_3^{(j)} = \frac{d\lambda_j^2 + de - ah}{ka - fd + a\lambda_j^2}$$

$$z_2^{(j)} = -(\lambda_j^2 + e + f z_3^{(j)})/a\lambda_j \quad ; \; j = 1,6$$

The seventh and eighth characteristic root is simply given as

$$\lambda_7 = (-q)^{\frac{1}{2}}$$

$$\lambda_8 = -(-q)^{\frac{1}{2}}$$

and the corresponding components of the characteristic vectors are

$$z_4^{(7)} = 1$$

$$z_4^{(8)} = 1$$

All other components $z_i^{(j)}$ are zero.

Now, let

$$M = \begin{pmatrix} \psi_i \\ \psi_{i,y} \end{pmatrix}$$

where

$$\psi_{i,y} = \sum_{j=1}^{8} p_j \lambda_j z_i^{(j)} e^{\lambda_j (y-y_e)} \quad ;i=1,4$$

so

$$m_i = \sum_{j=1}^{8} p_j n_i^{(j)} e^{\lambda_j (y-y_e)} \quad ;i=1;8$$

where

$$N = \begin{pmatrix} z_i^{(j)} \\ \lambda_j z_i^{(j)} \end{pmatrix} \quad ;i=1,4$$

The arbitrary constants p_j are obtained by inverting 8x8 matrix N,

$$P = N^{-1} M$$

The asymptotic boundary conditions require that P vanish whenever $Re(\lambda_j)>0$. This results in four equations of the form

$$(ED + F) \psi = 0$$

where both E and F are 4x4 matrices.

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Table 1.- Characteristics of the sample test cases on a 35° swept wing (chord, c = 8 ft).

Case No.	Streamwise Location, x/c	R	M	α	β	$\theta^{\dagger}$
1	0.001868	145	0.386	0.272117	-0.29181	51.96°
2	0.4639	3615	1.058	0.1153	0.0	26.93°
3	0.8921	1754	0.736	0.2432	-0.2654	34.80°

† Angle formed by the local potential flow vector with the x-axis.

Table 2.-Timings for the global eigenvalue method (time given in seconds on a CYBER 175 computer).

N	8th order system	6th order system
15	3.15	2.05
20	7.12	5.17
25	13.47	8.65

Table 3 Timings for the local eigenvalue method

N	8th order system	6th order system
20	0.61	0.40
40	1.14	0.77
80	2.22	1.52

Table 4.-Effect of the initial guess on the convergence of the local eigenvalue method for case 3 using N = 80, (8th-order system).

Initial guess	Time required for convergence
(0.011595,0.0023522)*	1.82
(0.01,0.002)	2.38
(0.02,0.004)	2.95
(0.02,0.008)	3.10
(0.005,0.001)	3.11
(0.03,0.006)	3.50
(0.04,0.004)	not converged

* equals the converged value

Table 5.-Computed group velocities for cases 1-3 using N = 80, (8th-order system).

Case No.	ω_α		ω_β	
	Central difference formula	Present method	Central difference formula	Present method
1	(0.66039, -0.00625)	(0.66036, -0.00636)	(0.73769 0.01386)	(0.73771, 0.01398)
2	(0.39893, 0.02105)	(0.39876, 0.02063)	(0.21884, -0.000987)	(0.21867, -0.00155)
3	(0.53507, -0.05212)	(0.53512, -0.05219)	(0.44301, -0.04472)	(0.44295, -0.04464)

Table 6.-Richardson extrapolation of the most unstable eigenvalue for case 1

Re(ω) x 10

i	N	m= 0	1	2	3	4	5
1	20	-.2686117	-.2629801	-.2630158	-.2630098	-.2630087	-.2630100
2	40	-.2643880	-.2630118	-.2630102	-.2630087	-.2630099	
3	60	-2.636234	-.2630106	-.2630089	0.2630099		
4	80	-.2633553	-.2630093	-.2630097			
5	120	-.2631631	-.2630096				
6	160	-.2630959					

Im(ω) x 100

i	N	m= 0	1	2	3	4	5
1	20	.6073290	.6191717	.6186855	.6185509	.6186694	.6186523
2	40	.6162110	.6187395	.6185593	.6186661	.6186525	
3	60	.6176158	.6186043	.6186542	.6186534		
4	80	.6180483	.6186418	.6186535			
5	120	.6183780	.6186506				
6	160	.6184972					

Table 7.- Richardson extrapolation of the most unstable eigenvalue for case 2

Re(ω) x 10

i	N	m= 0	1	2	3	4	5
1	20	.3940779	.3877680	.3889774	.3888798	.3888478	.3888521
2	40	.3893454	.3888430	.3888859	.3888486	.3888520	
3	60	.3890663	.3888751	.3888528	.3888518		
4	80	.3889827	.3888584	.3888519			
5	120	.3889136	.3888535				
6	160	.3888873					

Im(ω) x 100

i	N	m= 0	1	2	3	4	5
1	20	.0725901	.1230929	.1228928	.1227987	.1228500	.1228501
2	40	.1104672	.1229150	.1228046	.1228486	.1228501	
3	60	.1173826	.1228322	.1228437	.1228500		
4	80	.1197668	.1228408	.1228491			
5	120	.1214746	.1228470				
6	160	.1220750					

Table 8.-Richardson extrapolation of the most unstable eigenvalue for case 3

				Re(ω) x 10			
i	N	m= 0	1	2	3	4	5
1	20	.1171532	.1158833	.1158746	.1158764	.1158752	.1158759
2	40	.1162008	.1158756	.1158763	.1158752	.1158759	
3	60	.1160201	.1158761	.1158754	.1158758		
4	80	.1159571	.1158756	.1158758			
5	120	.1159118	.1158757				
6	160	.1158960					

				Im(ω) x 100			
i	N	m= 0	1	2	3	4	5
1	20	.2200906	.2362895	.2362199	.2362203	.2362156	.2362186
2	40	.2322398	.2362276	.2362203	.2362158	.2362185	
3	60	.2344553	.2362221	.2362163	.2362184		
4	80	.2352283	.2362177	.2362181			
5	120	.2357780	.2362180				
6	160	.2359705					

Table 9.-Comparison of eigenvalues obtained using different orders of extrapolation

Case No.	Three point extrapolation using N = 20, 30 and 40	Six point extrapolation (see tables 6-8)
1	(-0.026303, 0.0061848)	(-0.026301, 0.0061865)
2	(0.038889, 0.0012324)	(0.038885, 0.0012285)
3	(0.011587, 0.0023623)	(0.011587, 0.0023621)

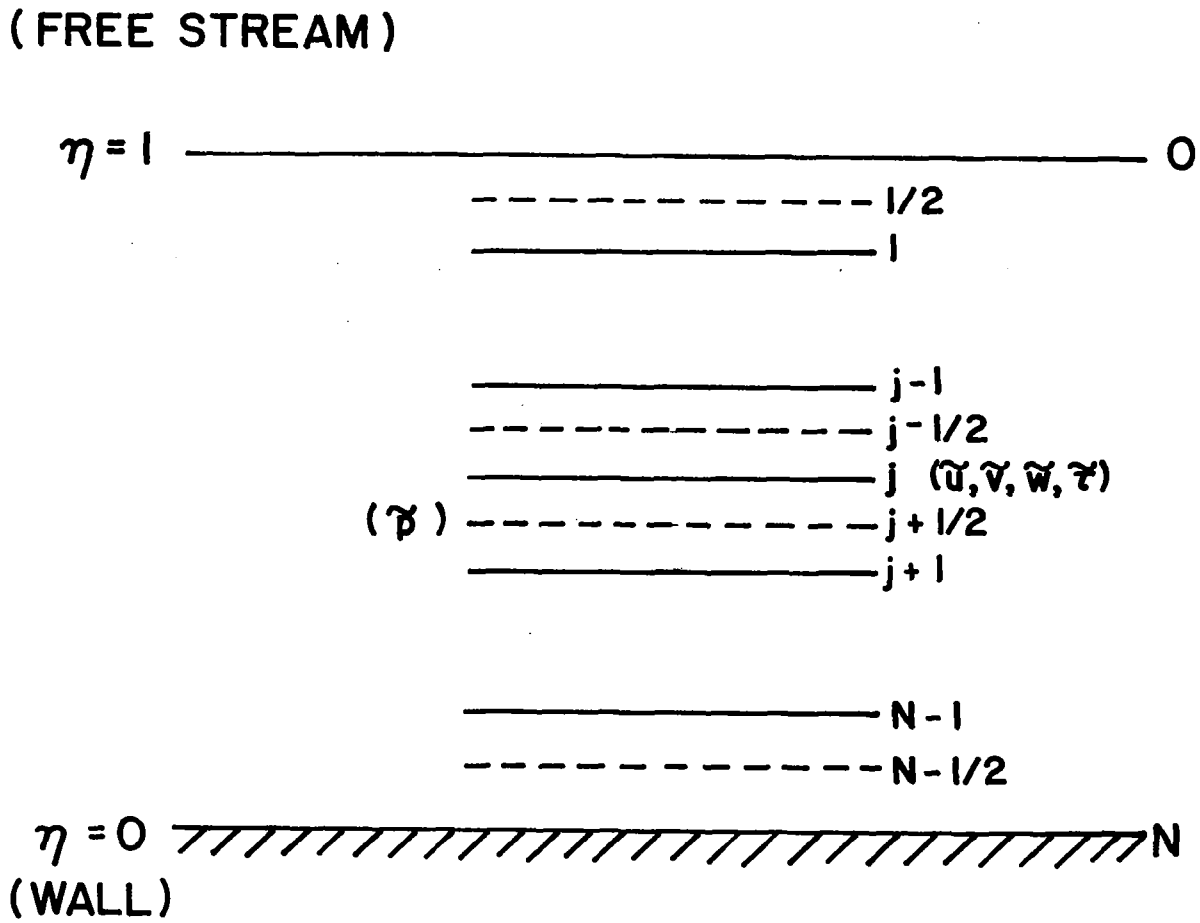

Figure 1.- A SCHEMATIC OF THE STAGGARD FINITE-DIFFERENCE GRID

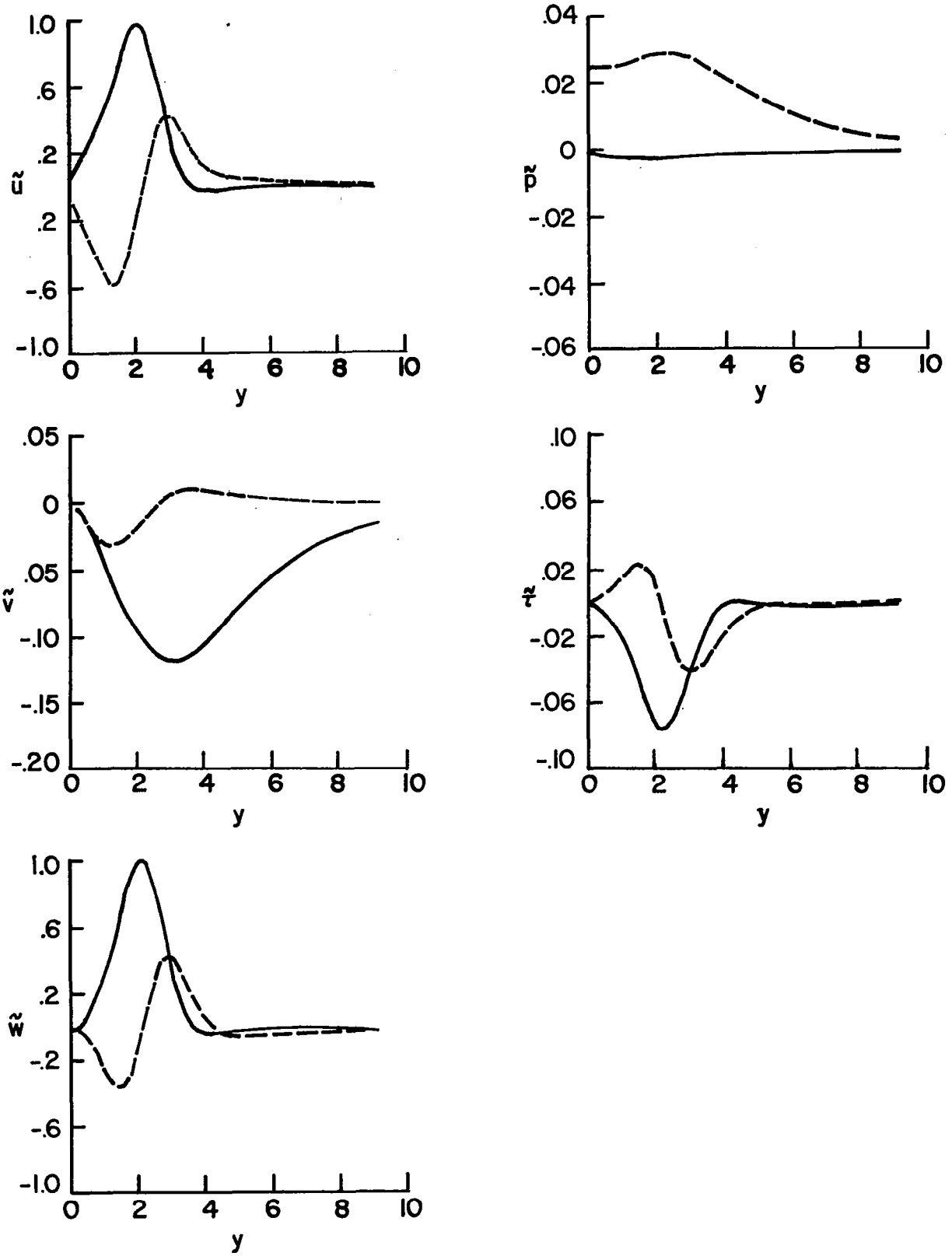

Figure 2.- COMPLEX DISTURBANCE AMPLITUDES PLOTTED VERSUS Y FOR CASE 1, —— REAL ---- IMAGINARY. THESE RESULTS WERE OBTAINED USING N = 80.

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Final Report

16. Abstract

A compressible stability analysis computer code is developed. The code uses a matrix finite-difference method for local eigenvalue solution when a good guess for the eigenvalue is available and is significantly more computationally efficient than the commonly used initial-value approach. The local eigenvalue search procedure also results in eigenfunctions and, at little extra work, group velocities. A globally convergent eigenvalue procedure is also developed which may be used when no guess for the eigenvalue is available. The global problem is formulated in such a way that no unstable spurious modes appear so that the method is suitable for use in a black-box stability code. Sample stability calculations are presented for the boundary layer profiles of an LFC swept wing.

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